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A DUALITY THEOREM FOR PLASTIC TORSION

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Abstract—Limit analysis of prismatic torsion bars was the earliest attempt to apply plasticity theory to a continuum. The simplicity of the problem made it feasible to use the two-dimensional Prandtl stress function, defined for the elastic torsion problems, for the plastic stress distributions as well. The gradient of the stress functions for plastic torsion has a constant magnitude, and hence a function of this type assumes the profile of a sand hill. This sand hill analogy of Nadai (1950, *The Theory of Flow and Fracture of Solids*, McGraw-Hill, U.K.) gave a visual sense of possible non-smoothness of such stress functions and thus discontinuous stress fields. Many stress functions of plastic torsion for relatively simple cross-sections have been constructed graphically. However, collapse modes in terms of warping functions were much less reported. In this paper, we shall establish a duality theorem which relates the correct stress function to the correct warping function, thus providing the means to obtain complete static and kinematic solutions. This dual variational principle leads naturally to a general numerical algorithm which guarantees convergence and accuracy. In this paper, we shall only present three exact solutions to verify the theorem, to demonstrate the possible non-smooth feature of the solutions and to reiterate this effective dual variational approach to limit analysis in general.

INTRODUCTION

The theory of perfect plasticity (Prager and Hodge, 1951) has long been applied to analyze the limit behavior of structures by Hodge (1959). As modern technology pushes for greater performance of material ductility and optimal structural design, limit analysis is now studied with resurgent interest. Designs for earthquake-resistant buildings and bridges, collision-safe automobiles, accident-tolerant nuclear installations, and light-weight peripheral equipment to match the lightning speed of computer output all need the help of limit analysis. However, the methodology and the theoretical foundation of limit analysis known to engineers are still at the 1960 level. The issue today is not just the capability of a solution. We need accuracy, efficiency and automation of computer software that can handle large problems with many design parameters. Furthermore, repeated computations under factorial growth of parameter combinations must be accomplished within a reasonable time to practically achieve optimal designs. To support the development of limit analysis and optimal design software, there must be sound mathematical analysis.

The theory of limit analysis has a deceptive simplicity. In fact, its mathematical structure is still being investigated in the recent research of functional analysis (Strang and Temam, 1980; Demengel, 1984; Teman, 1985), and modern calculus of variations (Cesari *et al.*, 1988). Functions involved in limit analysis are often non-smooth. Their derivatives must be interpreted in a generalized sense presented by Clarke (1983). To analyze the deeper aspects of these fine points may require advanced tools of mathematics, too technical for engineers. In this paper, we shall avoid arguments of technicality and use elementary mathematical language and physical intuition to enhance our understanding of certain abstract results.

Duality theorems and their applications to plastic analysis of plates, plane strain and plane stress problems have been presented by Yang (1987), Liu and Yang (1989) and Huh and Yang (1991) in connection with weak solutions of a variational integral called the virtual work. A variational principle for plastic torsion is presented here in the same light.

We first state the basic assumptions for the plastic torsion problem. A prismatic bar of arbitrary cross-section is made from a ductile material which may harden under plastic deformation but has an asymptotic behavior (perfect plasticity in the limit) in its stress–strain relation. The plastic behavior described here is called asymptotically perfect. The elastic property of the material, although not required in the analysis, is not explicitly excluded in the constitutive inequalities. We only assume a large elastic modulus (several orders greater than the asymptotic yield strength) so that elastic strains are implicitly neglected in comparison with plastic strains and the overall deformation remains small before the impending plastic collapse of the bar. Therefore, up to the point of collapse, a Lagrangian reference frame can be used to describe the motion and equilibrium about the undeformed configuration. This departure from the classical rigid–perfectly plastic model does not change the nature of limit analysis but will broaden its applicability.

The primal (or natural) formulation leads directly to statements that seek the greatest lower bound attained by the exact static solution. A variational procedure develops the dual formulation which minimizes a sharp upper bound functional in terms of admissible kinematic functions. A duality theorem equates the least upper bound to the greatest lower bound. We may choose to maximize the lower bound functional or minimize the upper bound functional to obtain the limiting torque (collapse load). Solving the primal–dual problems simultaneously produces complete static and kinematic solutions.

The sand hill analogy of Nadai (1950) made it simple to construct stress functions graphically, or by actually using sand to carry out the analogy if the cross-section is relatively simple. However, in either method, no accurate means of evaluating the limiting torque from the sand hill was ever developed. As an improvement, a half-analytic half-computational method developed by Yang (1979) successfully derived the ridge line equations of the sand hills for several non-simple cross-sections. Then a special finite element integration method produced limiting torques accurately. For kinematic solutions in terms of warping functions, only a line integration method of Mandel (1946) is available for simple cross-sections.

In this paper, we return to the fundamentals and concern ourselves with the duality theorem for the plastic torsion problems. It is the modern approach to all limit analysis problems. We shall first present the primal formulation in terms of the Prandtl stress function. The Cauchy–Schwartz inequality, when applied to the weak equilibrium equation (virtual work), leads the way to the dual formulation and the duality theorem. Three exact solutions are presented to demonstrate this duality. Based on this theorem, a general algorithm has been developed and successfully applied to problems of complex domains and difficult boundary conditions. This algorithm applies also to a non-linear wave problem (Osher and Sethian, 1988), thus furnishing another new flame propagation analogy (Yang, 1991) to the plastic torsion problems.

THE PRIMAL FORMULATION

The symmetric 3×3 matrix function of the form

$$\sigma = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{bmatrix}, \tag{1}$$

whose non-zero elements are functions of x and y in a domain D, represents a shear stress distribution in a cross-section of a prismatic torsion bar which extends between $0 \leqslant z \leqslant l$ where l is the length of the bar. By assuming an identical stress distribution in every cross-section, we may therefore regard D as a typical cross-section of the three-dimensional domain of the bar. It is sufficient to consider only the two stress components σ_{zx} and σ_{zy} which form a vector function $\sigma \in R^2(D)$. This is only an expediency to condense writing. We shall return to the three-dimensional domain, $D^3 = D \times [0, l]$, of the bar when a description calls for it.

A specific vector function $\sigma(x,y)$ is regarded as a point in the space $R^2(D)$. The statically admissible set S in that space consists of those points that satisfy the equilibrium equation and static boundary conditions. In the absence of body forces, we write the non-trivial equilibrium equation

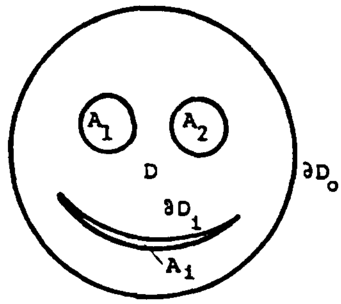

Fig. 1. A general cross-section of a prismatic torsion bar.

$$\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}=0, \tag{2}$$

to be satisfied in the domain D, and the stress-free boundary condition

$$\sigma_{zx}n_x+\sigma_{zy}n_y=0 \tag{3}$$

on the boundary ∂D which is representative of the entire lateral surface of the bar, where n_x and n_y are the non-trivial components of its outward normal vector. If the domain D is multiply connected, then the external boundary is denoted by ∂D_0 and the boundaries of the n holes are denoted by ∂D_i, $i = 1, 2, \ldots, n$ and $\partial D = \cup_0^n \partial D_i$. A general cross-section with holes is shown in Fig. 1.

The static boundary condition at two end surfaces, $z = 0, l$, is expressed in the integral form

$$T=\int_D (x\sigma_{zy}-y\sigma_{zx})\,\mathrm{d}A, \tag{4}$$

where T is the applied torque. All stresses that satisfy (2), (3) and (4) form a set $S \in R^2(D)$.

For a ductile material with the property of asymptotically perfect plasticity, the static equilibrium of the bar can be maintained with negligible deformation for sufficiently small T. As the value of T increases, the static equilibrium will eventually break down as the bar continues to deform with non-increasing torque. The condition for static admissibility under the greatest value of T is also the condition for an impending failure (collapse). This ductile failure also depends on the stress-bearing capacity of the material. A pointwise constitutive law may be expressed by the asymptotic yield criterion,

$$\|\sigma\|_2=\sqrt{\sigma_{zx}^2+\sigma_{zy}^2}\leqslant k, \tag{5}$$

where the Euclidean magnitude of the stress vector is bounded by the material constant k, the asymptotic yield stress in shear. This pointwise condition, when applied to the entire domain, forms a set $C \subset R^2(D)$ whose elements are called constitutively admissible.

The mathematical problem of seeking the greatest value, sup $T(\sigma)$, $\sigma \in L = S \cap C$ is defined as the primal problem. In engineering notation, we write

$$\begin{aligned}
&\textit{maximize} && T(\sigma)=\int_D (x\sigma_{zy}-y\sigma_{zx})\,\mathrm{d}A \\
&\textit{subject to} && \frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}=0 \quad \textit{in} \quad D, \\
& && \sigma_{zx}n_x+\sigma_{zy}n_y=0 \quad \textit{on} \quad \partial D, \\
&\textit{and} && \|\sigma\|_2\leqslant k.
\end{aligned} \tag{6}$$

Let $T^* = \sup T(\sigma)$, called the limiting torque. Then all the admissible stress distributions in D which correspond to $T \leqslant T^*$ form the set L of the lower bound solutions. Since S and C are convex sets, the set L is convex. It is non-empty since $\sigma = 0$ is in the set. It is also bounded since C is bounded. The supremum of $T(\sigma)$ taken over a convex, non-empty and bounded set necessarily exists.

Introducing the Prandtl stress function $\phi(x, y)$ such that

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \sigma_{zy} = -\frac{\partial \phi}{\partial x}, \tag{7}$$

the equilibrium eqn (2) is automatically satisfied by any such stress function. The primal problem (6) expressed in terms of this stress function is given by Yang (1979):

$$\begin{aligned} &\textit{maximize} && T(\phi(x,y)) \\ &\textit{subject to} && T = 2\int_D \phi(x,y)\,\mathrm{d}A + 2\sum_{i=1}^{n} \phi_i A_i, \\ & && \phi = 0 \text{ on } \partial D_0, \quad \phi = \phi_i \text{ on } \partial D_i, \quad i = 1, 2, \ldots, n, \\ &\textit{and} && \|\nabla\phi\|_2 \leqslant k, \end{aligned} \tag{8}$$

where ϕ_i is the undetermined constant value of ϕ on the internal boundary ∂D_i, A_i is the cross-sectional area of the ith hole and $\nabla\phi$ is the gradient vector of $\phi(x, y)$. The solution to problem (8) determines $T^* = T_{max}$ and $\phi^*(x, y)$ in D, including ∂D.

It has been shown that the maximization of (8) will drive the yield criterion (5) to its upper bound such that $\|\nabla\phi\|_2 = k$ for every point in D. The optimal solution $\phi^*(x, y)$ is a surface of constant absolute gradient. For a simply connected domain D, such a surface can be made by building a sand hill on a horizontal platform of D. Under gravity, sand will slide under a constant slope. Thus the profile of the sand hill will assume a surface of constant absolute gradient. This sand hill analogy becomes a less manageable set-up when D has holes. Worse yet, there exists no convenient method of integrating the sand hill volume to obtain the limiting torque T^*.

DUALITY

A standard tool in functional analysis (Royden, 1988), is the upper or lower bounding of a functional by another functional. For instance, instead of seeking the maximum of the original functional, we seek its least upper bound (the supremum). By minimizing an upper bound functional, we may recover the maximum of the original functional. This method of analysis will succeed only if the inequality relating the original functional to the upper bound functional is sharp such that the equality is inclusive. A functional which is bounded above has a supremum. A functional which is bounded below has an infemum. The theorem that equates the least upper bound to the maximum or the greatest lower bound to the minimum of a functional is called duality.

We begin with the weak equilibrium equation involving the original 3×3 stress matrix σ,

$$\int_{D^3} \mathbf{u}^t(\nabla \cdot \sigma)\,\mathrm{d}V = 0, \tag{9}$$

where t transposes a vector, $\nabla \cdot$ is the divergence operator in R^3 and the vector valued function $\mathbf{u} = (-\theta zy, \theta zx, \theta\psi(x, y))^t$ is chosen with a constant θ and an arbitrary warping function $\psi(x, y)$ in the set $K \subset R(D)$ to be defined as the set of kinematically admissible functions.

Integrating (9) by parts under a general divergence theorem and using the boundary conditions (3) and (4), we obtain after some simplification

$$T = \int_D \left[\sigma_{zx}\left(\frac{\partial \psi}{\partial x} - y\right) + \phi_{zy}\left(\frac{\partial \psi}{\partial y} + x\right)\right] \mathrm{d}A, \tag{10}$$

where the integrand may be regarded as the inner product of the stress vector $\sigma \in R^2(D)$ and the vector valued function called the strain rate,

$$\varepsilon = \begin{pmatrix} \dfrac{\partial \psi}{\partial x} - y \\ \dfrac{\partial \psi}{\partial y} + x \end{pmatrix}, \tag{11}$$

defined in the domain D. Using the Cauchy–Schwarz inequality and the constitutive inequality (5), we may rewrite (10) as

$$T = \int_D \sigma' \varepsilon \, \mathrm{d}A \leqslant \int_D \|\sigma\|_2 \|\varepsilon\|_2 \, \mathrm{d}A \leqslant k \int_D \|\varepsilon\|_2 \, \mathrm{d}A = \bar{T} \tag{12}$$

where a sharp upper bound functional $\bar{T}(\psi(x, y)) \geqslant T(\phi(x, y))$ is obtained.

The dual formulation, which seeks the least upper bound by solving

$$\textit{minimize} \quad k \int_D \sqrt{\left(\frac{\partial \psi}{\partial x} - y\right)^2 + \left(\frac{\partial \psi}{\partial y} + x\right)^2} \, \mathrm{d}A, \tag{13}$$

is a standard calculus of variation problem without constraint. Cesari *et al.* (1988) have recently proved that the absolute minimum of such an integral exists. The sharpness of the inequalities in (12) and the existence of the absolute minimum of (13) constitute the proof of the duality theorem for the plastic torsion problem:

$$\min_{\psi \in K} \bar{T}(\psi(x, y)) = T^* = \max_{\phi \in L} T(\phi(x, y)) \tag{14}$$

where T^* is the limiting torque. The elements of K and L are continuous functions defined in D. The functions in L must satisfy the constraint conditions in (8) while the functions in K have no constraint. Unlike the other limit analysis problems where the integrands involved in the integral functionals may contain unbounded but integrable measures, the torsion problems are devoid of such complications. Any ordinary numerical method, combining discretization and optimization, can be applied to the primal problem (8) and the dual problem (13). In this paper, we are concerned only with some exact solutions to demonstrate the theoretical aspects presented. The correct pair of a stress function and a warping function constitute the complete solution and give the unique limiting torque.

EXAMPLES

Three examples including bars of circular, square and rectangular cross-section are presented in this section. Although the solutions to these problems are well known, the purpose here is to demonstrate the duality theorem which is not. Numerical solutions of complicated cross-sections computed by a new algorithm based on the duality theorem are presented by Yang (1991).

Since these three cross-sections are simply connected we may rewrite (8) in a simpler form,

$$\begin{aligned} &\textit{maximize} \quad T = 2\int_D \phi(x,y)\,\mathrm{d}A, \\ &\textit{subject to} \quad \phi = 0 \text{ on } \partial D, \\ &\textit{and} \quad \|\nabla\phi\|_2 \leqslant k. \end{aligned} \tag{15}$$

For the circular bar of radius a, the solution of (15) is a cone. In a polar coordinate,

$$\phi^*(r) = k(a-r) \quad 0 \leqslant r \leqslant a, \tag{16}$$

which has a constant gradient $-k$; equals zero on the boundary; and yields the limiting torque $T^* = \frac{2}{3}\pi k a^3$.

By axisymmetry, no warping of the circular cross-section should be expected. Hence $\psi^* \equiv 0$ and from (13), we obtain

$$\bar{T}(\psi^*) = k\int_D \sqrt{x^2+y^2}\,\mathrm{d}A = \tfrac{2}{3}\pi k a^3 = T^* \tag{17}$$

and confirm the duality theorem (14).

Although the square and rectangular bars belong to the same family, different aspect ratios of the cross-sections cause great changes in warping functions. We shall treat these bars separately.

First, consider a $2a \times 2a$ square bar whose plastic stress function is a pyramid with a height ka on the square base. It is easy to verify that this pyramid function satisfies all the constraints in (15). The corresponding torque equals twice the volume of the pyramid,

$$T(\phi_{\text{pyramid}}) = \tfrac{8}{3}ka^3. \tag{18}$$

A trial warping function is chosen as

$$\psi(x,y) = [x\,\text{sign}\,(y) - y\,\text{sign}\,(x)]\,\min\{|x|,|y|\}, \tag{19}$$

which is obviously continuous in D. Substituting (19) into (13) and observing the symmetry of the integrand, the integration needs be carried out only in one-eighth of the domain D. We choose the sub-domain $\{(x,y): 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a,\ x \geqslant y\}$ and obtain

$$\bar{T}(\psi_{\text{trial}}) = 8k\left[\int_0^a \int_y^a 2(x-y)\,\mathrm{d}x\,\mathrm{d}y\right] = \tfrac{8}{3}ka^3. \tag{20}$$

We have indeed found the optimal solutions $\phi^* = \phi_{\text{pyramid}}$, $\psi^* = \psi_{\text{trial}}$ and verified the duality

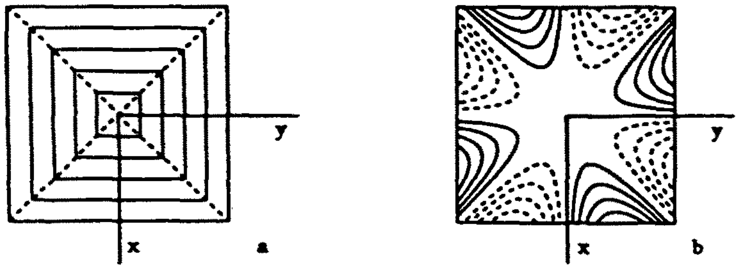

Fig. 2. The contours of stress and warping functions for a $2a \times 2a$ torsion bar.

$T(\phi^*) = \bar{T}(\psi^*) = 8ka^3/3$. The contour maps of $\phi^*(x, y)$ and $\psi^*(x, y)$ are shown in Fig. 2a, b.

Where the solid contours have positive values, the dashed ψ contours are negative and the dashed lines in the map of ϕ are ridge lines where the derivatives of ϕ are discontinuous. Since the warping function can be scaled by an arbitrary constant, the absolute values of the contours are immaterial.

Consider now a torsion bar with a $4a \times 2a$ cross-section. It is well known that the exact stress function for the rectangular bars is in the form of a roof function, as shown in Fig. 3a. For this particular bar, the volume under the roof can be calculated to give the limiting torque, $T^* = 20ka^3/3$.

Again, using the symmetry of the problem, the integration in (13) needs be carried out only in the first quadrant. We further divide the first quadrant, $[0, 2a] \times [0, a]$, into three sub-domains

$$
\begin{aligned}
D_1 &= \{(x, y) : x - a \geqslant y\}, \\
D_2 &= \{(x, y) : 0 \leqslant x - a \leqslant y\}, \\
D_3 &= \{(x, y) : 0 \leqslant x \leqslant a\}.
\end{aligned}
\tag{21}
$$

The warping function in the first quadrant is defined by

$$
\psi(x, y) = \begin{cases} y(x - y - 2a), & \text{in } D_1 \\ (x - a)^2 - xy, & \text{in } D_2 \\ -xy, & \text{in } D_3 \end{cases}
\tag{22}
$$

which is continuous across the boundaries of D_1, D_2 and D_3. Its symmetric extension is continuous in D. Substituting (22) into (13) and multiplying the result by four to cover the entire domain D, we obtain

$$
\bar{T}(\psi) = \tfrac{20}{3} ka^3
\tag{23}
$$

which agrees with the limiting torque calculated earlier from the roof function. Hence the roof function and the warping function defined in (22) are the correct pair of static and kinematic solutions. The contour maps of these two functions are shown in Fig. 3a, b. The (x, y) coordinates are rotated from the usual orientation so that the maps fit well in the width of the page.

There is a marked change in the warping function from the square to this rectangular domain. In the square ($2a \times 2a$) domain, ψ^* alternates its sign eight times as a point travels along a closed, convex curve around the origin. In the $4a \times 2a$ domain, the sign changes only four times. The transition can be seen in a $3a \times 2a$ domain for which the contour maps are shown in Fig. 4a, b, where two pairs of warping waves near the short edges of the rectangle first shrink then disappear as the aspect ratio increases.

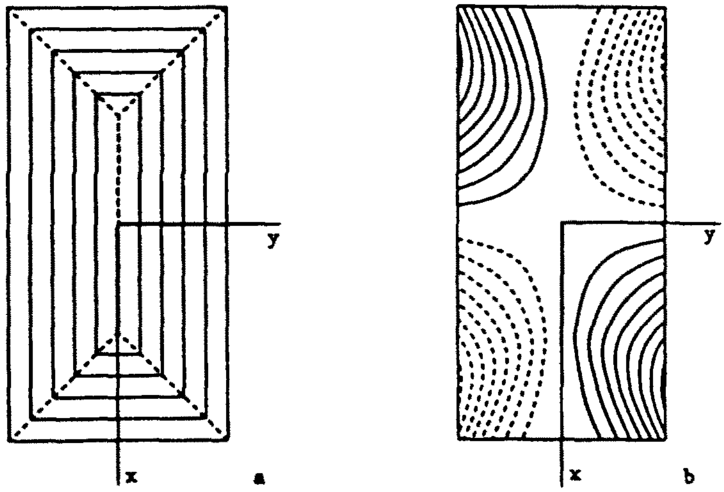

Fig. 3. The contours of stress and warping functions for a $4a \times 2a$ torsion bar.

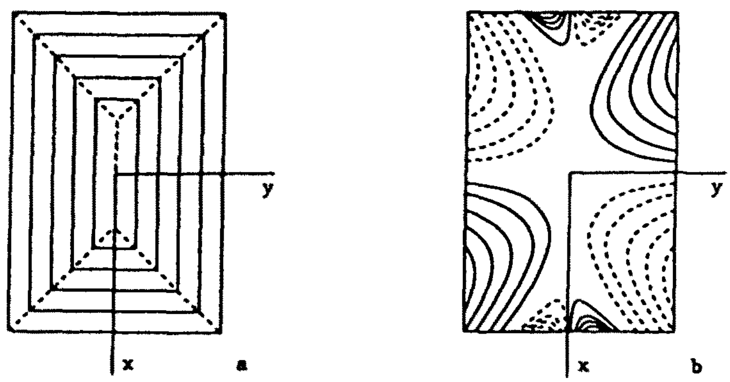

Fig. 4. The contours of stress and warping functions for a $3a \times 2a$ torsion bar.

FINAL REMARKS

There seems to be a misconception in the mechanics community that the issue of general upper and lower bound theorems for limit analysis has long been closed. In fact the proof of a duality theorem in plasticity remains an open topic in functional analysis. When one is presented, it usually entails the technical and sometimes new language of modern mathematics. The following scenario is not uncommon. A mathematician's highly technical proof and an engineer's highly intuitive solution of a plasticity problem met with polite silence or superficial communication in mixed company. Yet each needs the other's deeper insight to advance the state of science for the non-one-to-one, non-linear, and non-smooth problems encountered in the mathematical theory of plasticity, as well as in engineering applications.

A duality theorem under very broad proposition does not solve all the problems in special cases. The smoothness of a constitutive model, the boundary shape and loading all have a bearing on the non-smoothness and lack of uniqueness of the limit solutions. Each sub-class of problems such as plate, plane strain and plane stress, etc. possesses special characteristics and merits an independent study. We have a general proof of the duality theorem for plastic torsion problems. Three concrete examples with exact static and kinematic solutions have verified the abstract results and demonstrated the non-smooth nature of the solutions.

Since the stress and strain rate components are related to the derivatives of a stress function and a warping function respectively, these derivatives of non-smooth but continuous functions have finite jumps. Hence the stress and strain rate in plasticity belong to a class of functions called BV [bounded variation, Volpert (1967)] by mathematicians. The BV functions are studied under the deeper topics of functional analysis and calculus of variations as in the references cited. Engineers have produced many relatively simple BV solutions in plasticity and called them kinematically admissible (the class K mentioned earlier). The definition of K has been rather vague in the engineering literature. We intend to use the results in modern functional analysis to help define K precisely for each class of problems in plasticity. Identifying the correct function space is vitally important in approximations such as the finite element and the finite difference methods. For plastic torsion problems, the stress functions and warping functions are absolutely continuous and their first partial derivatives belong to BV.

For a stress distribution $\sigma \in$ BV, how does it satisfy the differential equation of equilibrium (2) when the derivatives of the stress components along a ridge line are unbounded? The answer lies in either of the following two interpretations: the unbounded derivatives in (2) are equal but opposite in sign so they add up to zero. The other interpretation is the standard weak (integral) equation such that any finite element of D is in equilibrium.

Although this paper is intended only as an exposition for the theoretical aspects of the problem considered, the duality theorem (14) is a fundamental basis for good numerical algorithms. The unique optimality of the primal–dual problems and convergence of an iterative algorithm can be obtained by the closing of the duality gap. However, the uniqueness of the static and kinematic solutions cannot be guaranteed. This can also be understood from an engineering viewpoint that the collapse modes may not be unique in reality.

REFERENCES

Cesari, L., Brandi, P. and Salvadori, A. (1988). Existence theorems for multiple integrals of the calculus of variations for discontinuous solutions. *Annali. Mat. Pura Appl.* **152**(4), 95–121.

Clarke, F. H. (1983). *Optimization and Nonsmooth Analysis*. John Wiley, New York.

Demengel, F. (1984). Fonctions à hessian borné. *Ann. Inst. Fourier, Grenoble* **34**(2), 155–190.

Hodge, P. G. (1959). *Plastic Analysis of Structures*. McGraw-Hill, Maidenhead, U.K.

Huh, H. and Yang, W. H. (1991). A general algorithm for limit solutions of plane stress problems. To appear in *Int. J. Solids Structures*.

Liu, K. H. and Yang, W. H. (1989). A finite element method for limit analysis of plane strain extrusions through square dies. *Num. Meth. Indus. For. Proc. NUMIFORM 89* (Edited by Thompson *et al.*), pp. 191–196.

Mandel, J. (1946). Sur les déformations de la torsion plastique. *C.R. Acad. Sci., Paris* **222**, 1205–1207.

Nadai, A. (1950). *The Theory of Flow and Fracture of Solids*. McGraw-Hill, Maidenhead, U.K.

Osher, S. and Sethian, J. A. (1988). Fronts propagating with curvature dependent speed: algorithm based on Hamilton–Jacobi formulations. *J. Comp. Phys.* **79**, 12–49.

Prager, W. and Hodge, P. G. (1951). *Theory of Perfectly Plastic Solids*. John Wiley, New York.

Royden, H. L. (1988). *Real Analysis*, 3rd edn. Macmillan, New York.

Strang, G. and Temam, R. (1980). Duality and relation in the variational problems of plasticity. *J. Mécanique* **19**, 493–572.

Temam, R. (1985). *Mathematical Problems in Plasticity*. Gauthier-Villars, Paris.

Volpert, A. L. (1967). The spaces BV and quasi-linear equations. *Mat. sb.* **73**, 226–267.

Yang, W. H. (1979). A practical method for limit torsion problems. *Comp. Meth. Appl. Mech. Engng* **19**, 151–158.

Yang, W. H. (1987). A duality theorem for plastic plates. *ACTA Mechanica* **69**, 177–193.

Yang, W. H. (1991). Flame propagation analogy for plastic torsion, to appear.